# § 5.5 Max Noether's Fundamental Theorem

Intersection cycle $F \cdot G := \sum_{P \in \mathbb{P}^2} I(P, F \cap G)\, P$.

zero-cycle on $\mathbb{P}^n$ is a formal sum $\sum_{P \in \mathbb{P}^2} n_P P$

$$\{\text{zero-cycle}\} = \bigoplus_{P \in \mathbb{P}^n} \mathbb{Z}$$

almost all vanish

$n_P \in \mathbb{Z}$

- deg $\left(\sum n_P P\right) := \sum n_P$
- $\sum n_P P \geq 0 \overset{\text{def}}{\Longleftrightarrow} n_P \geq 0 \quad \forall P \in \mathbb{P}^2$
- $\sum n_P P \geq \sum m_P P \overset{\text{def}}{\Longleftrightarrow} n_P \geq m_P \ \forall P$
- Intersection cycle $F \cdot G := \sum_{P \in \mathbb{P}^2} I(P, F \cap G)\, P$.

Fact (Bézout's thm) $F \cdot G$ is a positive zero-cycle of degree $mn$

5.5.2 Question: When $\exists B$ s.t. $B \cdot F = H \cdot F - G \cdot F$ ?

($\Leftarrow$ $H \equiv BG \bmod F$)

Noether's conditions are satisfied at $P$ w.r.t. $F, G, H$. if

$$H_* \in (F_*, G_*) \triangleleft \mathcal{O}_P(\mathbb{P}^2)$$

Thm (max Noether's fundamental theorem) $F, G, H$ = proj. plane curves $\gcd(F, G) = 1$. Then

$\exists A, B$ s.t. $H = AF + BG \Leftrightarrow$ noether's conditions are satisfied at each $P \in F \cap G$.

pf: $\Rightarrow$) $H = AF + BG \Rightarrow H_* = A_* F_* + B_* G_*$ at $\forall P \Rightarrow \checkmark$

$\Leftarrow$) WMA: $V(F, G, Z) = \phi$

$F_* = F(X,Y,1)$, $G_* = G(X,Y,1)$, $H_* = H(X,Y,1)$

noether's condition $\Rightarrow$ $H_* = 0 \in \mathcal{O}_P(\mathbb{P}^2)/(F_*, G_*)$

§2.9. Prop 6 $\Rightarrow$ $H_* = 0 \in k[X,Y]/(F_*, G_*)$

$\Rightarrow H_* = aF_* + bG_*$ $a, b \in k[X,Y]$.

$\Rightarrow Z^r H = AF + BG$ for some $r, A, B$.

$\Rightarrow H = A'F + B'G$ $\left( k[X,Y,Z]/(F,G) \xrightarrow{\cdot Z} k[X,Y,Z]/(F,G) \right)$

$\Rightarrow H = A'_S F + B'_* G$

criteria that noether's conditions holds $G | H \pmod F$

Prop 1. $F, G, H$ = plane curves $P \in F \cap G$. Noether's conditions holds at $P$ if any of the following are true.

1) $F$ & $G$ meet transversally at $P$ and $P \in H$

2) $P$ simple on $F$ & $I(P, H \cap F) \geq I(P, G \cap F)$

3) $F$ & $G$ has distinct tangents at $P$ and

$$m_P(H) \geq m_P(F) + m_P(G) - 1$$

Pf: (2). $P$ simple $\Rightarrow \mathcal{O}_P(F) = \mathrm{DVR} \Rightarrow \mathrm{ord}_P^F$

$$I(P, H\cap F) \geq I(P, G\cap F) \Rightarrow \mathrm{ord}_P^F(H) \geq \mathrm{ord}_P^F(G)$$

$$\Rightarrow \overline{H}_* \in (\overline{G}_*) \triangleleft \mathcal{O}_P(F)$$

$$\Rightarrow \overline{H}_* = 0 \in \mathcal{O}_P(F)/(\overline{G}_*) = \mathcal{O}_P(\mathbb{P}^2)/(F_*, G_*)$$

(3). WMA: $P = [0:0:1]$ & $m_P(H_*) \geq m_P(F_*) + m_P(G_*) - 1$

$$H_* \subseteq I^{m+n-1} \underset{\substack{\Uparrow \\ \text{have distinct tangents}}}{\subseteq} (F_*, G_*) \subseteq \mathcal{O}_P(\mathbb{P}^2)$$

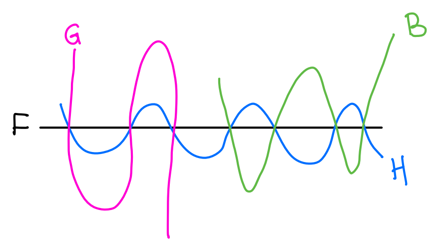
(1). (3) $\Rightarrow$ (1) or (2) $\Rightarrow$ (1).

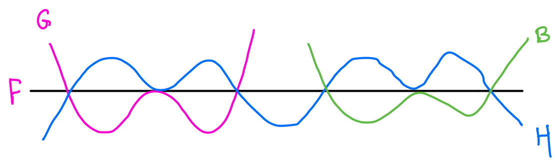
**Cor** If either $\qquad \Rightarrow F\cdot G \leq F\cdot H$

1) $\# F\cap G = \deg F \cdot \deg G$ & $F \cap G \subseteq H$. or

2) $F\cap G$ simple on $F$ & $H\cdot F \geq G\cdot F$, then

$$\exists B \text{ s.t. } B\cdot F = H\cdot F - G\cdot F$$



## § 5.6 Applications of Noether's Theorem.

A few of interesting consequences

**Prop 2:** $C, C' =$ cubics. $Q =$ conic

$$C' \cdot C = \sum_{i=1}^{9} P_i \quad \& \quad Q \cdot C = \sum_{i=1}^{6} P_i$$

Then $P_7, P_8, P_9$ lie on a straight.

pf: $F = C$, $G = Q$, $H = C'$ in (2) of cor. $\square$

**Cor 1 (Pascal).** if a hexagon is inscribed in an irr. conic, then the opposite sided meet in collinear points.

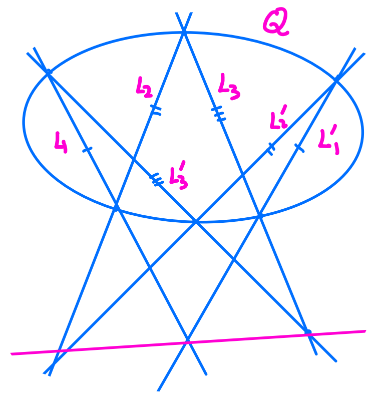
pf:

$C = L_1 L_2 L_3$

$C' = L_1' L_2' L_3'$

$G = Q$

Prop 2 $\Rightarrow$ ✓

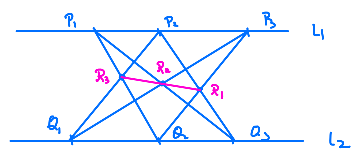



**Cor 2. (Pappus)** $L_1, L_2 =$ line $P_1, P_2, P_3 \in L_1$, $Q_1, Q_2, Q_3 \in L_2$

$L_{ij} = \overline{P_i Q_j}$

$R_k = L_{ij} \cap L_{ji} \quad \forall \{i, j, k\} = \{1\ 2\ 3\}$

$\Rightarrow R_1, R_2, R_3 =$ collinear.



**Prop 3.** $C$ = irr. cubic , $C', C''$ = cubics

$$C'C = \sum_{i=1}^{9} P_i \quad (\text{simple on } C\text{, may not distinct})$$

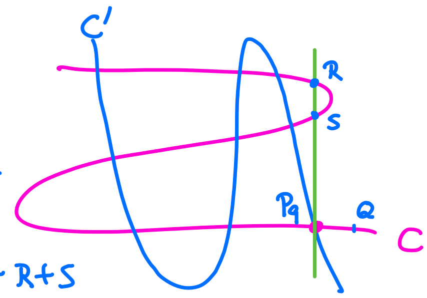
$$C''\cdot C = \sum_{i=1}^{8} P_i + Q \Rightarrow Q = P_9$$

pf: $L\cdot C = P_9 + R + S$

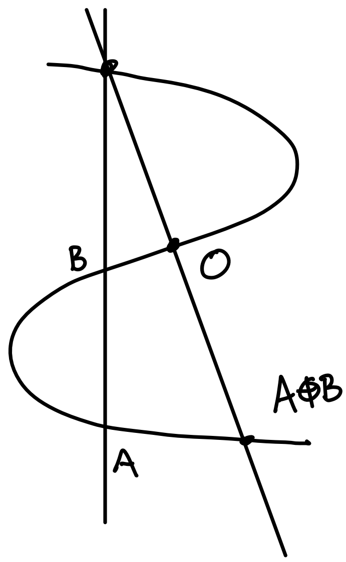
$LC''\cdot C = C'\cdot C + Q + R + S$

$\Rightarrow \exists L'$ s.t. $L'\cdot C = Q+R+S$

$\Rightarrow L = L' \Rightarrow P_9 = Q$.




Addition on a cubic:




**Prop 4.** $(C, \oplus)$ forms an abelian gp with $O$ being the identity.

pf: Only associativity difficult:

<u>$P + Q + R$</u>

$C' = l_1 l_2 l_3$ $\quad C'' = M_1 M_2 M_3$

prop 3 $\Rightarrow T' = T''$